

Construction of potentials using mixed scattering data*

M. Lassaut^{a,b}, S.Y. Larsen^c, S.A. Sofianos^b, J.C. Wallet^d

^a *Institut de Physique Nucléaire*

CNRS/Université Paris-Sud 11 (UMR8608), F-91406 Orsay Cedex, France

^b *Physics Department, University of South Africa, Pretoria 0003, South Africa*

^c *Department of Physics, Temple University, Philadelphia, PA 19122, USA*

^d *Laboratoire de Physique Théorique, Bât. 210*

CNRS/Université Paris-Sud 11 (UMR8627), F-91405 Orsay Cedex, France

Abstract

The long-standing problem of constructing a potential from mixed scattering data is discussed. We first consider the fixed- ℓ inverse scattering problem. We show that the zeros of the regular solution of the Schrödinger equation, $r_n(E)$ which are monotonic functions of the energy, determine a unique potential when the domain of energy is such that the $r_n(E)$'s range from zero to infinity. The latter method is applied to the domain $\{E \geq E_0, \ell = \ell_0\} \cup \{E = E_0, \ell \geq \ell_0\}$ for which the zeros of the regular solution are monotonic in both parts of the domain and still range from zero to infinity. Our analysis suggests that a unique potential can be obtained from the mixed scattering data $\{\delta(\ell_0, k), k \geq k_0\} \cup \{\delta(\ell, k_0), \ell \geq \ell_0\}$ provided that certain integrability conditions required for the fixed ℓ -problem, are fulfilled. The uniqueness is demonstrated using the JWKB approximation.

*e-mail: lassaut@ipno.in2p3.fr.

1 Introduction

Three-dimensional (3-D) inverse quantum scattering problem amounts to reconstruct some interaction potential from a set of experimentally accessible scattering data. For spherically symmetric potentials¹ $V(r)$, the case that will be considered throughout this paper, it is known [1] that the 3-D problem $(-\Delta_{\vec{r}} + V(r))\psi(E, \vec{r}) = E\psi(E, \vec{r})$ ($\Delta_{\vec{r}}$ is the 3-D Laplacian, E is the energy) can be reduced, thanks to the partial wave decomposition $\psi(E, \vec{r}) = \sum_{l \in \mathbb{N}} \sum_{m=-l}^{m=l} \frac{\psi_l(E, r)}{r} Y_{l,m}(\theta, \phi)$ (the $Y_{l,m}$'s are the spherical harmonic functions) to the study of the radial Schrödinger equation with centrifugal barrier term depending on the angular momentum ℓ . This latter² is given by

$$\left(\frac{d^2}{dr^2} + E - V(r) - \frac{(\ell + 1/2)^2 - 1/4}{r^2} \right) \psi_\ell(E, r) = 0 \quad (1.1)$$

where $\psi_\ell(E, r)$ is called the regular solution which is uniquely defined, as usual [1, 2], by the Cauchy condition $\lim_{r \rightarrow 0} \psi_\ell(E, r)r^{-\ell-1} = 1$. It behaves for positive values of E as $\psi_\ell \propto \sin(kr - \ell\pi/2 + \delta(\ell, k))$ when $r \rightarrow \infty$ ($k = \sqrt{E}$, see second footnote), provided that $V(r)$ satisfies the integrability condition [2]

$$\int_b^{+\infty} |V(r)|dr < \infty, \quad b > 0, \quad \int_0^\infty r|V(r)|dr < \infty \quad (1.2)$$

Here, the $\delta(\ell, k)$'s are the phase shifts, the relevant scattering data for the present paper. Notice that the whole scheme can be extended to real values for ℓ verifying $2\ell + 1 > 0$ [1].

Basically, approaches to 3-D inverse scattering can be classified in two categories [1, 2], namely, those using scattering data for fixed values of either the energy E or the angular momentum ℓ . In the first category, the so-called fixed- E inverse scattering problem, Loeffel [3] obtained theorems predicting a unique potential from the knowledge of the phase-shifts $\delta(\ell, k)$ at a specific energy $E = k^2$, for all (non-discrete) non negative values of $\lambda = \ell + 1/2$. When the set of data is reduced to discrete values of $\lambda = \ell + 1/2$ for non-negative integer ℓ , Carlson's theorem [4] predicts a unique potential $V(r)$, provided that this latter belongs to a suitable class [2, 3]. The Newton series permits one to construct the potential $V(r)$ from the knowledge of the discrete partial wave scattering data [3]. In the second category, the so-called fixed- ℓ inverse scattering problem, the potential $V(r)$ that satisfies (1.2) can be constructed from the phase-shifts $\delta(\ell, k)$ given for all momenta $k \in (0, \infty)$ and from the discrete spectrum data (bound state energies and the corresponding normalization constants). For details, see [2].

Apart from the above two distinct approaches there exists also a possibility of constructing a potential starting from input scattering data that are partly E -dependent and partly ℓ -dependent. The idea of exploiting simultaneously E - and ℓ -dependent scattering data was first explored by Grosse and Martin [5] for the construction of confining potentials. They conjectured that the knowledge of the ground state energies $E_\ell^{(0)}$, for all non-negative integers ℓ , allows one to recover the potential in a unique way. The problem of reconstructing potentials from bound states has been studied numerically in Ref.

¹Notations: $\vec{r} = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$, $r > 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$.

²The potential $v(r)$ is normalised such that $\frac{2m}{\hbar^2}V(r) = v(r)$ where m is the mass for the particle. We use throughout this paper the units [2] $\hbar = 2m = 1$; then, $V(r) = v(r)$ and the energy E verifies $E = k^2$ where k is the wave number.

[6]. In [7], Rudyak and Zakhariev have constructed potentials from E - and ℓ -dependent data, however assuming that $aE + b\ell(\ell + 1)$ is a constant. In this case, the corresponding analysis is based on an extension of the Newton's method.

In this paper, we construct a potential obtained from input scattering data that are partly E - and ℓ -dependent, when no extension of the Newton's method is available. Let us summarize the results of our study. First we show that, in the fixed- ℓ problem, the knowledge of the zeros of the regular solution the Schrödinger equation permits one to determine the potential. More precisely, we show that the zeros of the regular solution, denoted hereafter by $r_n(E)$, $n \geq 1$, which are known to be monotonic functions of the energy, whatever the behavior of the potential may be [8], determine a unique potential provided the domain of the energy is such that $r_n(E)$ range from 0 to ∞ . Note that the reconstruction of the potential from the zeros of the regular solution enters the category of inverse nodal problems introduced by Hald and McLaughlin [9]. As a second result, we show that piecewise constant potentials can be constructed from only one single line of zeros still ranging from 0 to ∞ , say $r_{n_0}(E)$, for some fixed n_0 . As a third result of this paper, we prove uniqueness theorems, stating basically that two different potentials satisfying (1.2) cannot have a common line of zeros ranging from 0 to ∞ . In the second part of the analysis, we turn to the mixed ℓ and E problem and apply the above results to the domain $\{E \geq E_0, \ell = \ell_0\} \cup \{E = E_0, \ell \geq \ell_0\}$ for which the zeros of the regular solution for the Schrödinger equation are monotonic functions on both parts of this domain and still range from zero to infinity. This domain cannot be dealt with (any extension of) the Newton's method. The first part of the analysis suggests that a unique ℓ - and E -independent potential could be obtained from the set of mixed scattering data given by $\{\delta(\ell_0, k), k \geq k_0\} \cup \{\delta(\ell, k_0), \ell \geq \ell_0\}$. As the last result of this paper, we show that this is indeed true within the semi-classical JWKB approximation, provided there is only one turning point. Note that the uniqueness theorem 1 still applies for the mixed data, leading to the construction of piecewise constant potentials from the zeros of the regular solution of the Schrödinger equation. General details on the JWKB approximation and its full applicability can be found e.g in [10] while specific applications to the Schrödinger equation may be found e.g in [1, 2]. Basically, it provides a way to determine a semi-classical expansion (in \hbar) of the solution of the Schrödinger equation and is valid whenever the potential changes slowly during an oscillation of this solution.

Our analysis has natural applications in heavy ion physics as well as general properties of scattering processes although applications in other areas, such as acoustics and/or geophysics seem likely to be possible. In heavy ion (elastic) scattering physics, a natural application of the present paper concerns the problem of discrete ambiguities encountered in optical model analysis [11] and would at least provide a new insight on past works of this problem [12, 13]. We have undertaken the corresponding study which will be reported elsewhere [14]. The present analysis can be easily extended to more complicated set of (mixed) scattering data. In particular, we emphasize that the most noticeable feature of our approach is that it can be applied to inverse scattering problems not reducible to the $\ell = 0$ -fixed inverse scattering problem through a Liouville transformation (defined for instance in [2]), as it is the case for the fixed- E problem for potentials of finite range [3] or for the case treated by Rudyak and Zakhariev [7] also for potentials of finite range. Note that, when the inverse scattering problem is reducible to the usual s -wave inverse scattering problem, there exists an extension of the Newton's method available for the

scattering problem considered.

The paper is organised as follows. The section 2 involves the whole analysis. In subsection 2.1, we consider the fixed- ℓ problem. In subsection 2.2, we discuss in detail the mixed scattering problem. The subsection 2.3 deals with the JWKB approximation applied to the mixed scattering data. In section 3, we discuss the results and conclude.

2 Formalism

2.1 The fixed- ℓ inverse scheme

We assume that the potential fulfills the integrability condition (1.2). Let us consider the regular solution of the Schrödinger equation $\psi_\ell(E, r)$ for fixed positive energy E and fixed ℓ , a function of $r \geq 0$. It can be realized that $\psi_\ell(E, r)$ and $\psi'_\ell(E, r)$ cannot vanish at the same value(s) of r for $2\ell + 1 > 0$ (except for $r = 0$) and $\ell > 0$ as shown in [15] so that $\psi_\ell(E, r)$ can only have simple zeros (except at $r = 0$). Owing to the asymptotic properties of $\psi_\ell(E, r)$ and $\frac{\psi_\ell}{\psi'_\ell}(E, r)$ as $r \rightarrow \infty$, it can be further realized that the set of zeros is countable. According to these observations, these zeros (the origin $r = 0$ being excluded) can then be naturally ordered starting from the smallest value that will be denoted by $r_1(\ell, E)$. A generic zero of the regular solution will be denoted by $r_n(\ell, E)$, $n \in \mathbb{N}$, $n \geq 1$. This defines the numbering of the zeros that is used in the following analysis. Notice that, thanks to the fact that all the zeros are simple, one has the following (strict) ordering $0 < r_1(\ell, E) < r_2(\ell, E) < \dots < r_n(\ell, E) < \dots$.

For any potential, the zeros satisfy the monotonicity properties, i) $E = k^2 \mapsto r_n(\ell, E)$ is a decreasing function, as has been shown by Sturm in 1830's [8] and ii) $\ell \mapsto r_n(\ell, E)$ is an increasing function. This can be easily shown. More precisely, we have

$$\frac{\partial}{\partial E} r_n(\ell, E) = - \int_0^{r_n(\ell, E)} dr' \psi_\ell(E, r')^2 / \left(\frac{\partial}{\partial r} \psi_\ell(E, r_n(\ell, E)) \right)^2 \quad (2.1)$$

$$\frac{\partial}{\partial \ell} r_n(\ell, E) = (2\ell + 1) \int_0^{r_n(\ell, E)} dr' \frac{\psi_\ell(E, r')^2}{r'^2} / \left(\frac{\partial}{\partial r} \psi_\ell(E, r_n(\ell, E)) \right)^2 \quad (2.2)$$

Note that, as mentioned above, $\psi_\ell(E, r)$ and $(\partial \psi_\ell / \partial r)(E, r)$ cannot vanish simultaneously for $2\ell + 1 > 0$, except for $r = 0$ and $\ell > 0$ [15], so that the denominators in (2.1) and (2.2) cannot vanish.

For potentials satisfying ((1.2)) the function $E \mapsto r_n(\ell, E)$ is such that $r_n(\ell, E) \rightarrow 0$ for $E \rightarrow \infty$ [16]. This is a consequence of equation (I.5.6) of [2]. In the absence of bound states $r_n(\ell, E) \rightarrow \infty$ for $E \rightarrow 0$.

For negative or zero values of E , the regular solution of the Schrödinger equation has zeros provided that the potential has bound states, and that $E > E_1$, where $E_1 < 0$ denotes the ground state energy³. The number of zeros is finite for values of the energy E such that $E_1 < E \leq 0$. For details see [16]. When the potential has N bound states, $E_1 < E_2 \dots < E_N$, if $n \leq N$, we have $r_n(\ell, E) \rightarrow \infty$ for $E \rightarrow E_n$ [16], whereas for $n > N$, $r_n(\ell, E) \rightarrow \infty$ as $E \rightarrow 0$. This is illustrated in the Fig. 1 where we have drawn the first four zeros of the regular solution for a Bargmann transparent potential of Ref. [17], which

³We adopt this notation to denote the ground state by E_1 instead of the traditionally used E_0 in order to be consistent with the meaning given to n , namely to denote zeros of the wave function

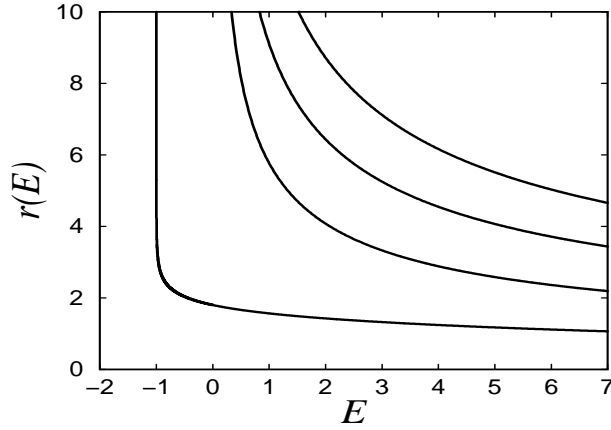


Figure 1: Zeros $r_n(E)$, $n = 1, 2, 3$, and 4, of the regular s-wave solution for a Bargmann potential (for more details see text).

has a bound state at the energy $E = -1$ in $1/L^2$ units. All zeros, but the first one, go to infinity as E goes to 0. The first zero, however, goes to infinity when $E \rightarrow E_1 = -1$. In the fixed- ℓ inversion, if all the zeros are assumed to be known, i.e. $E \mapsto r_n(\ell, E)$ is known and $r_n(\ell, E)$ describes the entire interval $[0, \infty[$, then the potential $V_\ell(r)$ is uniquely determined. This can be easily checked. For this we consider the Sturm-Liouville problem on $[0, R]$, i.e., the equation

$$\psi_\ell''(r) + \left(E - V_\ell(R - r) - \frac{\ell(\ell + 1)}{(R - r)^2} \right) \psi_\ell(r) = 0, \quad (2.3)$$

coupled with the Dirichlet conditions

$$\psi_\ell(0) = \psi_\ell(R) = 0.$$

The spectral data are the eigenvalues E_n^* 's, such that $R = r_n(\ell, E_n^*)$, and the normalization constants [18]

$$\rho_n = \frac{\int_0^R \psi_\ell(r')^2 dr'}{\psi_\ell'(0)^2}.$$

The latter are given by the positive values

$$\rho_n = -\frac{d}{dE} r_n(\ell, E)|_{E=E_n^*}.$$

The potential, assumed to be square integrable on $[0, R]$ is uniquely determined on $[0, R]$ by the spectral data $\{E_n^*, \rho_n\}$ [19]. The technique of constructing V from this set of data is well-known [2, 18, 20, 21].

If we consider now the Sturm Liouville problem on $[0, R]$ consisting of the equation

$$\psi_\ell''(r) + \left(E - V_\ell(r) - \frac{\ell(\ell + 1)}{r^2} \right) \psi_\ell(r) = 0, \quad (2.4)$$

coupled with the Dirichlet conditions

$$\psi_\ell(0) = \psi_\ell(R) = 0$$

we recover the Gel'fand-Levitan approach at the limit $R \rightarrow \infty$ [2, 22] .

For the purpose of using mixed, E - and ℓ -dependent data, we start with the following remark. For a fixed ℓ , the n th zero of the regular solution can be considered as a function of energy $r_n(\ell, E)$, which is monotonic and such that $\psi_\ell(E, r_n(\ell, E)) \equiv 0$. It defines a line of zeros as we mentioned in the introduction. Moreover $E \mapsto r_n(\ell, E)$ admits an inverse function $r \mapsto E_{n,\ell}(r)$ which is also monotonic and is the inverse of this line of zeros. For example, consider the potential $V \equiv 0$ in the S-wave ($\ell = 0$). The regular solution is proportional to $\sin(\sqrt{E}r)$. The lines of zeros are given by $r_n(0, E) = n\pi/\sqrt{E}$ and the inverses of these lines are $E_{n,0} = n^2\pi^2/r^2$.

Suppose that the inverses $r \mapsto E_{n,\ell}$, of all lines of zeros $n \geq 1$, are known for r running from 0 to ∞ . This implies that $E_n^* = E_{n,\ell}(R)$ is also known and that the ρ_n are given by

$$\frac{1}{\rho_n} = -\frac{d}{dr}E_{n,\ell}(r)|_{r=R}$$

We can then construct the desired potential, assumed to be locally square integrable on $[0, R]$. If only the inverse $r \mapsto E_{n,\ell}$ of a single line of zeros is known, no method is available to recover the potential, except in the special case of piecewise constant potentials. What we can show is a uniqueness property, if $E_{n_1,\ell}(r)$ is known for all positive r (see the theorems given below).

In the special case of piecewise constant potentials, having discontinuities at values of $r = a_j$, $j = 1, \dots, j_{max}$, and being zero for $r > a_{j_{max}}$, it is easy to show that there is a one to one correspondence between the discontinuities of the third derivative of $E_{n_1,\ell}(r)$ with respect to r and the discontinuities of V . This suggests the following lemmas:

Lemma 1: *For a piecewise constant potential, the knowledge of a single line of zeros $r(E) = r_n(\ell, E)$ (n fixed) allows the reconstruction of the potential in a unique way provided that the latter function has values ranging from zero to infinity when $E \in]0, +\infty[$.*

The following variant of the lemma 1, although useless for the mixed set of data considered in the next subsection, works also for potentials having bound states. Denoting $r \mapsto E(r)$ the inverse function of the function $E \mapsto r(E)$ which is the single line of zeros $r(E)$ considered, we state that

Lemma 2: *For a piecewise constant potential, the knowledge of the inverse of a single line of zeros $E(r) = E_{n,\ell}(r)$ allows the reconstruction of the potential in a unique way provided that the latter function of the variable r is known on the whole interval $]0, +\infty[$. The proof uses the arguments given above, in particular the monotonicity property of the line with respect to the energy, and the third derivative of*

$$\psi(E(r), r) \equiv 0.$$

To be more specific, if E''' has a discontinuity at $r = a$ then

$$\frac{d^3 E}{dr^3}(a^+) - \frac{d^3 E}{dr^3}(a^-) = -2 \frac{dE}{dr}(a) [V(a^+) - V(a^-)] . \quad (2.5)$$

This is equivalent to the relation

$$\frac{d^3 r}{dE^3}(E_a^+) - \frac{d^3 r}{dE^3}(E_a^-) = 2 \left(\frac{dr}{dE}(E_a) \right)^3 [V(r(E_a^+)) - V(r(E_a^-))] \quad (2.6)$$

Since the potential is zero for $r > a_{j_{max}}$, the relation (2.5) allows us to reconstruct the potential between $a_{j_{max}}$ and $a_{j_{max}-1}$. The procedure can be repeated at each a_j , and the potential is obtained at $r \neq a_j$ by summing the successive values at each discontinuity appearing beyond r .

As an illustration consider the potential

$$V(r) = \begin{cases} -2 & r < 2 \\ -1 & 2 < r < 3 \\ 0 & r > 3 \end{cases} \quad (2.7)$$

In Fig. 2, we have drawn the function $r \mapsto -E'''(r)/(2E'(r))$, related to the inverse $E(r)$ of the first line of zeros for the S -wave regular solution of the Schrödinger equation with the potential (2.7). Clearly, the discontinuities of $r \mapsto -E'''(r)/(2E'(r))$ happen at the

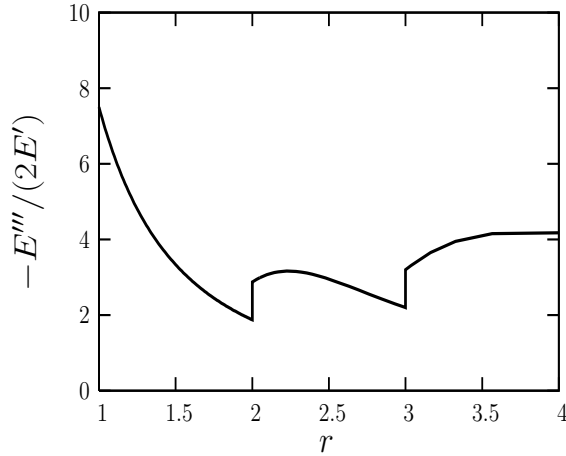


Figure 2: The structure of the function $-E'''(r)/(2E'(r))$ related to the inverse $E(r)$ of the first line of zeros for the S -wave regular solution corresponding to the potential (2.7).

points where V has discontinuities, namely $r = 2$ and $r = 3$, and the equation (2.5) is satisfied. We know that the potential is zero beyond $r = 3$. So we have $0 = V(3^+)$. From the curve the discontinuity $V(3^+) - V(3^-)$ is equal to 1 then $V(3^-) = -1$. As V is piecewise constant, it is constant on $]2, 3[$ and we have $V(2^+) = V(3^-) = -1$. From the curve the discontinuity $V(2^+) - V(2^-)$ is equal to 1 so that $V(2^-) = -2$. We thus recover the potential (2.7).

This method cannot be applied to a potential defined by a continuous function. Nevertheless, for such potentials, the following uniqueness theorems hold:

Theorem 1: *Consider two potentials V_1 and V_2 satisfying (1.2) and locally constant in the vicinity of zero. For the fixed- ℓ problem, assume that both potentials have no bound states. Suppose that two integers n_1 and n_2 exist such that the n_1 -th line of zeros for the regular solution for V_1 coincides with the n_2 -th line of zeros for the regular solution for V_2 , ($r_{n_1}(\ell, E) \equiv r_{n_2}(\ell, E)$) with both lines having values ranging from 0 to ∞ as the energy E varies in $]0, \infty[$. Then $V_1 \equiv V_2$.*

Its variant, still useless for our mixed set of data, but working for potentials having bound state yields:

Theorem 2: Consider two potentials V_1 and V_2 satisfying (1.2) and locally constant in the vicinity of zero. For the fixed- ℓ problem, assume that two integers n_1 and n_2 exist such that the inverse of the n_1 -th line of zeros for the regular solution for V_1 coincides with the inverse of the n_2 -th line of zeros for the regular solution for V_2 on the whole interval $]0, +\infty[$, $((\forall r > 0) E_{n_1, \ell}(r) \equiv E_{n_2, \ell}(r))$. Then $V_1 \equiv V_2$.

Let us prove the theorem 1. The proof of its variant, the theorem 2, quite similar to that of theorem 1, is not reported here.

First let $\psi_1(E, r)$ and $\psi_2(E, r)$ (we suppress the index ℓ for convenience) be the regular solutions of the Schrödinger equation for potentials V_1 and V_2 , respectively, constrained by the Cauchy conditions $\lim_{r \rightarrow 0} \psi_i(E, r) r^{-\ell-1} = 1$, $i = 1, 2$ in the vicinity of zero. For an S -wave, this is equivalent to $\psi'_i(E, 0) = 1$, $i = 1, 2$, the prime being the derivative with respect to r . Setting $\Delta V = V_1 - V_2$ and noting that the Wronskian is zero for $r = 0$ we obtain

$$\begin{aligned} W(\psi_1, \psi_2) &= \psi'_1(E, r)\psi_2(E, r) - \psi_1(E, r)\psi'_2(E, r) \\ &= \int_0^r dr' \Delta V(r') \psi_1(E, r') \psi_2(E, r') . \end{aligned} \quad (2.8)$$

For $r = r(E)$ (we adopt the convention $r_{n_1}(\ell, E) = r_{n_2}(\ell, E) = r(E)$) we have

$$(\forall E) \quad \int_0^{r(E)} dr' \Delta V(r') \psi_1(E, r') \psi_2(E, r') = 0 . \quad (2.9)$$

The latter integral exists since ψ_i , $i = 1, 2$ are continuous and both potential V_1 and V_2 satisfy (1.2)) and are locally constant in the vicinity of zero.

Differentiating twice Eq. (2.9) with respect to E , which is feasible as the first and the second derivative of $\psi_1(E, r)\psi_2(E, r)$ with respect to the energy E is a function continuous of the variable r [23], and taking into account that $\psi_i(E, r(E)) = 0$, $i = 1, 2$ for every value of E , we obtain

$$(\forall E) \quad \int_0^{r(E)} dr' \Delta V(r') \frac{\partial^2}{\partial E^2} [\psi_1(E, r') \psi_2(E, r')] = 0 \quad (2.10)$$

We recall that, due to its monotonicity, $E \mapsto r(E)$ admits an inverse $r \mapsto E(r)$. We define the Kernel

$$K(r, r') = \frac{\partial^2}{\partial E^2} [\psi_1(E(r), r') \psi_2(E(r), r')] , \quad r' \leq r . \quad (2.11)$$

The diagonal part, $K(r, r)$, equal to

$$K(r, r) = 2 \frac{\partial}{\partial E} \psi_1(E(r), r) \frac{\partial}{\partial E} \psi_2(E(r), r) = 2 \left(\frac{dr}{dE} \right)^2 \frac{\partial}{\partial r} \psi_1(E(r), r) \frac{\partial}{\partial r} \psi_2(E(r), r) . \quad (2.12)$$

is never zero, since ψ_i and $\frac{d}{dr} \psi_i$ ($i = 1, 2$) do not vanish simultaneously, except for $r = 0$ and for $\ell > 0$ [15].

Introducing the definition (2.11), the equation (2.10) is equivalent to

$$(\forall r \geq 0) \quad \int_0^r dr' K(r, r') \Delta V(r') = 0 . \quad (2.13)$$

Since the potential V_1 (V_2) is locally constant in the vicinity of zero, there exist \tilde{V}_1 and ϵ_1 (\tilde{V}_2 and ϵ_2) such that

$$\begin{aligned} (\forall r \leq \epsilon_1) \quad V_1(r) &= \tilde{V}_1 \\ (\forall r \leq \epsilon_2) \quad V_2(r) &= \tilde{V}_2 \end{aligned} \quad (2.14)$$

In the vicinity of zero, the n_1 -th line of zeros of the regular solution for V_1 is given by

$$r_{n_1}(\ell, E) = \frac{j_{\ell, n_1}}{\sqrt{E - \tilde{V}_1}} \quad (2.15)$$

where j_{ℓ, n_1} is the n_1 -th zero, except for the origin, of the function $x \mapsto j_\ell(x)$ involving the spherical Bessel function $j_\ell(x)$ [24].

This implies that

$$\frac{d \ln r_{n_1}(\ell, E)}{dE} = -\frac{1}{2} \frac{1}{E - \tilde{V}_1}. \quad (2.16)$$

Taking into account (2.14) and (2.16) as well as the identity $r_{n_1} \equiv r_{n_2}$ we see that for $r \leq \epsilon = \inf(\epsilon_1, \epsilon_2)$ we have $\tilde{V}_1 - \tilde{V}_2 = 0$. Therefore $\Delta V(r)$ is zero for $r \leq \epsilon$.

Differentiating (2.13) with respect to r , (possible since the derivative $r' \mapsto K_r(r, r')$ is continuous) and taking into account that ΔV is zero for $r \leq \epsilon$, leads to the Volterra equation

$$(\forall r \geq \epsilon) \quad \Delta V(r) + \int_{\epsilon}^r dr' K_1(r, r') \Delta V(r') = 0 \quad (2.17)$$

where

$$K_1(r, r') = \frac{K_r(r, r')}{K(r, r)}, \quad K_r(r, r') = \frac{\partial K}{\partial r}(r, r') \quad (2.18)$$

Let us consider (2.17) for $r \in [\epsilon, R]$. The Kernel K_1 is bounded and continuous. This is due to the fact that $K(r, r)$ is continuous and never vanishes on the compact interval $[\epsilon, R]$. So $|K(r, r)|$ is bounded from below by a strictly positive value. Thus Eq. (2.17) has a unique solution for $r \in [\epsilon, R]$ [23], which reads $\Delta V(r) = 0$. By increasing R we can conclude that $\Delta V(r) = 0$ for every r . This terminates the proof. Note that the fact that $K_1(r, r)$ behaves like $1/r$ in the vicinity of zero forbids us to conclude that (2.17) has a unique solution for $\epsilon = 0$ [25]. This is the reason why the potentials have been assumed to be locally constant in the vicinity of zero.

2.2 Inversion scheme from mixed data

Consider the set $\{E \geq E_0, \ell = \ell_0\} \cup \{E = E_0, \ell \geq \ell_0\}$. Similarly to Sect. 2.1, we may define lines of zeros in which the n -th line of zeros $r_n(\ell, E)$ of the regular solution describes a line formed of two parts. In the first part, the zeros $r_n(\ell_0, E)$ range from $r = 0$, ($E = \infty$), to $r_0 = r(\ell_0, E_0)$, ($E = E_0$), as the energy E varies from ∞ to E_0 (ℓ_0 being fixed); in the second part, $r_n(\ell, E_0)$ has values ranging from $r_0 = r(\ell_0, E_0)$, ($\ell = \ell_0$), to ∞ , ($\ell = \infty$) as ℓ ranges from ℓ_0 to ∞ (E_0 being fixed). This has been verified for potentials satisfying the integrability conditions (1.2) [2]. The monotonicity property required in the above lemma 1 and theorem 1 is preserved in both domains.

The lemma 1 is still valid for piecewise constant potentials. In the special case where a discontinuity $v = V(r_0^-) - V(r_0^+)$ appears at the junction point between both parts of the domain, i.e. at $r_0 = r_n(\ell_0, E_0)$, the value of v can be determined as follows. One first adds v to the bare value V_0 of the potential at the origin, obtained by adding all discontinuities (except v). Then, for r close to the origin, the zeros are given by

$$r_n(\ell_0, E) = \frac{j_{\ell_0, n}}{\sqrt{E - V_0 - v}} \quad (2.19)$$

and the value of v can be obtained from the relation

$$\frac{d \ln r_n(\ell_0, E)}{dE} = -\frac{1}{2} \frac{1}{E - V_0 - v} . \quad (2.20)$$

To demonstrate the applicability of the theorem 1, we consider potentials satisfying the integrability conditions (1.2) and locally constant in the vicinity of zero. Using the Wronskian relation (2.8), we have to show that, if

$$\int_0^{r(\ell_0, E)} dr' \Delta V(r') \psi_1(\ell_0, E, r') \psi_2(\ell_0, E, r') = 0 \quad \text{for } E \geq E_0 , \quad (2.21)$$

and

$$\int_0^{r(\ell, E_0)} dr' \Delta V(r') \psi_1(\ell, E_0, r') \psi_2(\ell, E_0, r') = 0 \quad \text{for } \ell \geq \ell_0 , \quad (2.22)$$

then $V_1 \equiv V_2$. (Here we have put as previously $r_{n_1} = r_{n_2} = r$.)

This can be proved in two steps. First, for $r \leq r_0 = r(\ell_0, E_0)$, Eq. (2.21) is zero for every $E \geq E_0$. Consequently, $V_1 \equiv V_2$ for $r \leq r_0$ when V_1, V_2 are locally constant in the vicinity of zero.

Secondly, taking into account the above argument, Eq. (2.22) can be rewritten

$$\int_{r(\ell_0, E_0)}^{r(\ell, E_0)} dr' \Delta V(r') \psi_1(\ell, E_0, r') \psi_2(\ell, E_0, r') = 0 \quad (2.23)$$

Differentiating twice with respect to ℓ , which is feasible as the first and the second derivative of $\psi_1(\ell, E_0, r) \psi_2(\ell, E_0, r)$ with respect to ℓ is continuous with respect to the variable r [23], and inverting the monotonic function $\ell \mapsto r(\ell, E_0)$, denoted by $r \mapsto \ell(r, E_0)$, we obtain

$$\begin{aligned} (\forall r \geq r_0) \quad & \int_{r_0}^r dr' \Delta V(r') K(r, r') = 0 \\ (\forall r' \leq r) \quad & K(r, r') = \frac{\partial^2}{\partial \ell^2} [\psi_1(\ell(r, E_0), E_0, r') \psi_2(\ell(r, E_0), E_0, r')] . \end{aligned} \quad (2.24)$$

The diagonal part $K(r, r)$ is equal to

$$\begin{aligned} K(r, r) &= 2 \frac{\partial}{\partial \ell} \psi_1(\ell(r, E_0), E_0, r) \frac{\partial}{\partial \ell} \psi_2(\ell(r, E_0), E_0, r) \\ &= 2 \left(\frac{dr}{d\ell} \right)^2 \frac{\partial}{\partial r} \psi_1(\ell(r, E_0), E_0, r) \frac{\partial}{\partial r} \psi_2(\ell(r, E_0), E_0, r) , \end{aligned} \quad (2.25)$$

and does not vanish for $r \geq r_0 > 0$.

Differentiating Eq. (2.24) with respect to r , (possible as $r' \mapsto K_r(r, r')$ is continuous) we obtain the Volterra equation

$$(\forall r \geq r_0) \quad \Delta V(r) + \int_{r_0}^r dr' K_1(r, r') \Delta V(r') = 0 \quad (2.26)$$

where K_1 is given by (2.18). Consider now Eq.(2.26) for $r \in [r_0, R]$. Since $K(r, r)$ in Eq. (2.25) is continuous on $[r_0, R]$, its absolute value reaches its strictly positive minimum m . The Kernel $K_1(r, r')$ is then bounded and continuous and the equation (2.26) has a unique solution $\Delta V(r) = 0$ on $[r_0, R]$. Increasing R , we conclude that $\Delta V(r) = 0$ for every $r \geq r_0$ and then $V_1 \equiv V_2$ on the whole half axis.

We have shown that a single line of zeros, which, for the data considered, always has values ranging from zero to infinity and moreover is monotonic, determines the potential uniquely. The remaining question is to examine whether the set of mixed data

$$\{\delta(\ell = \ell_0, k) \mid k \in [k_0, +\infty[\} \cup \{\delta(\ell, k_0) \mid \ell \in [\ell_0, +\infty[\} \quad (2.27)$$

associated to the set $\{\ell = \ell_0, k \in [k_0, \infty[\} \cup \{k = k_0, \ell \in [\ell_0, \infty[\}$, determines a line of zeros, and thus the potential, in a unique way - which is suggested by the analogy with the ℓ -fixed problem. In the absence of bound states, all lines of zeros $E \mapsto r_n(\ell, E)$, monotonic with respect to E , range from zero (E infinite) to infinity ($E = 0$) when E has values ranging from infinity to zero. In this case, we know that the potential, when it satisfies (1.2), is recovered in a unique way, given the phase-shifts $\delta(\ell, k)$ for all $k \geq 0$. The condition (1.2) excludes all pathologies; for example, potentials behaving asymptotically like $1/r^2$, encountered in particular in the presence of a zero energy bound state, or ghost components in the Jost function [17]. With the mixed data we are in the same situation, namely all the lines of zeros are monotonic and range from zero (E infinite, $\ell = \ell_0$) to infinity ($E = E_0$, ℓ infinite). Thus we expect that (2.27) is associated with a unique potential decreasing faster than $1/r^2$ at infinity. We cannot prove this in the general case, but we can investigate the problem in a JWKB approach.

Note that a way to treat the general case is to set a self-consistent procedure. First we note that, in order to calculate the phase-shift say $\delta(\ell, k)$, there is no need to know the potential for distances smaller than $r_n(\ell, E)$, whatever is n . It is sufficient to consider the Schrödinger equation on the interval $[r_n(\ell, E), \infty[$ because only the ratio ψ_ℓ/ψ'_ℓ , where the prime denotes the derivative with respect to r , has to be known to determine the phase-shift. In this respect, we first determine the potential in terms of the phase-shift $\delta(\ell, k_0)$ ($\delta(\ell_0, k)$) and the n -th (n fixed for all the procedure) zero of the regular solution by solving the Schrödinger equation on $[r_n(\ell, k_0), \infty[$ ($[r_n(\ell_0, k), \infty[$) because the phase-shift does not depend on the potential $V(r)$ for $r \leq r_n(\ell, k_0)$ ($r_n(\ell_0, k)$). The algebraic construction of the potential is guaranteed by the fact that the monotonicity property of the n -th line of zeros of the regular solution in terms of the energy does not depend on the potential. Then we have to determine the zeros. For instance take the zeros of the free solution $V \equiv 0$ then determine V then recalculate the zeros etc..

We first consider results based on the Born approximation. In the seventies Reignier [26] used a Born approximation of the scattering amplitude to show that the knowledge of the phase-shift at a fixed energy, $E_0 = k_0^2$ say, for each integer ℓ is equivalent to the

knowledge of the Fourier sine transform of the potential $rV(r)$,

$$g(q) = \int_0^\infty \sin(qr) rV(r) dr, \quad (2.28)$$

for $q \leq 2k_0$. The scattering amplitude is determined from the phase-shifts at fixed energy $\delta(\ell, k_0)$ for $\ell = 0, 1, 2, \dots$, $E = E_0 = k_0^2$.

Generally, the integral is assumed to be zero for $q > 2k_0$ [2, 12] leading to potentials

$$rV(r) = \frac{2}{\pi} \int_0^{2k_0} \sin(qr) g(q) dq, \quad (2.29)$$

such that $rV(r)$ is an entire function of r of order 1. Other extensions of $g(q)$ are studied in [27].

More recently Habashy and Wolf [28] have studied the reconstruction of a potential, having compact support and spherical symmetry, from its 3D-Fourier transform throughout the Ewald limiting sphere, $|k_0(\vec{u} - \vec{u}')|$ where \vec{u}, \vec{u}' are normalized to unity and take all possible directions. This is equivalent to the Fourier transform of the potential for values of \vec{k} such that $|\vec{k}| \leq 2k_0$. In this case, the inverse sine transform $r\tilde{V}(r)$ coming from Eq. (2.29) is given in terms of the potential $V(r)$ by

$$r\tilde{V}(r) = \int_{-R}^R dr' V(r') \frac{\sin[2k_0(r - r')]}{\pi(r - r')}, \quad -R \leq r \leq R, \quad (2.30)$$

where R denotes the support of V . For negative values of r , the authors of [28] have put $V(-r) = V(r)$. The latter equation can be solved numerically, thanks to the spectrum of the kernel involved in its right hand side, namely, to the $\nu_n \in]0, 1[$, $\psi_n \in L^2([-1, 1])$ (depending on k_0) such that

$$\int_{-1}^1 dx' \frac{\sin[2k_0(x - x')]}{\pi(x - x')} \psi_n(x') = \nu_n \psi_n(x), \quad -1 \leq x \leq 1. \quad (2.31)$$

Coming back to Eq. (2.29), a possible way to extend our knowledge of $g(q)$ beyond $2k_0$ is to take the Born approximation for the missing phase shifts $\delta(\ell = 0, k)$ for $k \geq k_0$. This is given by

$$\int_0^\infty \sin(kr)^2 V(r) dr = -k\delta(\ell = 0, k). \quad (2.32)$$

The derivative with respect to k yields

$$g(q) = \int_0^\infty \sin(qr) rV(r) dr = -\frac{d(k\delta(\ell = 0, k))}{dk}, \quad \forall q = 2k \geq 2k_0. \quad (2.33)$$

This implies that $g(q)$, known for $q \leq 2k_0$, is now known for every positive q , and that $V(r)$ is uniquely given by

$$rV(r) = \frac{2}{\pi} \int_0^\infty \sin(qr) g(q) dq. \quad (2.34)$$

Consequently, the knowledge of $\{\delta(\ell, k_0), \ell \in \mathcal{N}\} \cup \{\delta(\ell = 0, k), k \geq k_0\}$ allows us to recover the potential in Born approximation if k_0 is sufficiently high.

2.3 The JWKB approximation

In what follows, we examine to what extent the mixed data (2.27) allow us to recover the potential in the semi-classical JWKB approximation. General details on the JWKB approximation and its full applicability can be found e.g in [10] while specific applications to the Schrödinger equation may be found e.g in [1, 2]. It has also been extensively used in the fixed energy inverse scattering (see, for example, [29, 12, 13, 30] and references therein). Basically, we recall that this semi-classical approximation provides a way to determine a semi-classical expansion (in \hbar) of the solution of the Schrödinger equation and is valid whenever the potential changes slowly during an oscillation of this solution.

In the JWKB approximation, the phase-shift is given by

$$\delta(\ell, k) = \lim_{r \rightarrow \infty} \left(\int_{r(\lambda, k)}^r K(\lambda, k, r') dr' - \int_{r_{\text{free}}(\lambda, k)}^r K_{\text{free}}(\lambda, k, r') dr' \right) \quad (2.35)$$

(here $E = k^2$ in $1/L^2$ units and $\lambda = \ell + 1/2$). In the absence of a Coulomb component, we have

$$K(\lambda, k, r) = \sqrt{k^2 - V(r) - \lambda^2/r^2}, \quad K_{\text{free}}(\lambda, k, r) = \sqrt{k^2 - \lambda^2/r^2}. \quad (2.36)$$

In (2.35), $r(\lambda, k)$ is the turning point relative to the function K considered, assumed here to be unique for the sake of simplicity. More precisely $r(\lambda, k)$ is a solution of the equation $K(\lambda, k, r) = 0$.

For the part of the spectrum $\delta(\lambda, k_0)$, $\lambda \geq \lambda_0$ we use the results of Sabatier [12] and Cuer [13] for a single turning point. We assume that ($k = k_0$ being fixed) for every λ , the equation

$$k_0^2 - V(r) - \lambda^2/r^2 = 0 \quad (2.37)$$

has a unique solution denoted by $r(\lambda, k_0)$. This happens when the function $r \mapsto g(r) = r^2(k^2 - V(r))$ is monotonic. For $V \equiv 0$ the equation (2.37) has a unique solution, denoted by $r_{\text{free}}(\lambda, k) = \lambda/k$. In [12, 13] the phase-shift is given by:

$$\delta(\ell, k_0) = \int_{\lambda}^{\infty} d\lambda' \sqrt{\lambda'^2 - \lambda^2} \frac{d}{d\lambda'} \ln \left(\frac{r(\lambda', k_0)}{r_{\text{free}}(\lambda', k_0)} \right) \quad (2.38)$$

where $V(r)$ is assumed to be differentiable to ensure the derivative of the turning point with respect to λ exists. The function

$$\lambda \mapsto g(\lambda) = \lambda \frac{d}{d\lambda} \ln \left(\frac{r(\lambda, k_0)}{r_{\text{free}}(\lambda, k_0)} \right) \quad (2.39)$$

is assumed to be Lebesgue integrable.

From Eq. (2.38) we have

$$\frac{d}{d\lambda} \delta(\ell, k_0) = -\lambda \int_{\lambda}^{\infty} d\lambda' \frac{1}{\sqrt{\lambda'^2 - \lambda^2}} \frac{d}{d\lambda'} \ln \left(\frac{r(\lambda', k_0)}{r_{\text{free}}(\lambda', k_0)} \right) \quad (2.40)$$

the derivation being possible for $\lambda \neq 0$ as the integrand is dominated on $]\lambda, \infty[$ by the integrable function $\lambda' \mapsto |g(\lambda')|/(\lambda' \sqrt{\lambda'^2 - \lambda^2})$.

Using the Lebesgue-Fubini theorem [31, 32] which works when $d\delta(\ell, k_0)/d\lambda \in L^1(\mathcal{R})$ and bounded, as made for the Abel transform, the turning point is given in terms of the phase-shifts by

$$\ln \left(\frac{r(\lambda, k_0)}{r_{\text{free}}(\lambda, k_0)} \right) = \frac{2}{\pi} \int_{\lambda}^{+\infty} \frac{d\delta(\ell' = \lambda' - 1/2, k_0)}{d\lambda'} \frac{1}{\sqrt{\lambda'^2 - \lambda^2}} d\lambda' \quad (2.41)$$

provided that the logarithm is zero for λ infinite. The above hypothesis that $d\delta(\ell, k_0)/d\lambda$ is bounded avoids pathologies in the integration in the vicinity of λ . For a single turning point we recover the result of Loeffel: the knowledge of $\delta(\ell, k)$ for every $\lambda = \ell + 1/2$ positive allows us to recover the potential. Indeed, for λ infinite the logarithm tends to zero and $r(\lambda, k_0)$ tends to infinity. When λ tends to zero the equation (2.37) has as unique solution $r = 0$ when $\lim_{r \rightarrow 0} V(r)r^2 = 0$.

For $r \geq r_0 = r(\lambda_0, k_0)$, the potential is given by

$$V(r(\lambda, k_0)) = k_0^2 - \frac{\lambda^2}{r(\lambda, k_0)^2} = k_0^2 \left[1 - \left(\frac{r_{\text{free}}(\lambda, k_0)}{r(\lambda, k_0)} \right)^2 \right]. \quad (2.42)$$

Note that the behavior of the potential for $r(\lambda, k_0)$ or λ infinite is related to the behavior of the phase-shift $\delta(\ell, k_0)$ for λ infinite.

For $r \leq r_0$, we introduce the turning point $r = r(\lambda_0, k)$ as the solution of

$$k^2 - V(r) - \frac{\lambda_0^2}{r^2} = 0. \quad (2.43)$$

We assume that the latter equation has a unique solution. This happens when $r \mapsto h(r) = V(r) + \lambda_0^2/r^2$ is monotonic, for example for $V \geq 0$ or for V attractive but “weak” enough when compared to the centrifugal barrier, in the sense that $\|r^2 V'(r)\|_{\infty} < \lambda_0^2$. The aforementioned function h is positive and infinite at the origin and zero for $r = \infty$. Thus, we are sure to have at least one solution. The monotonicity of h implies the uniqueness of the solution of the equation $h(r) = k^2$, denoted by $r(\lambda_0, k)$. It implies also the monotonicity of $k \mapsto r(\lambda_0, k)$, which is monotonically decreasing with respect to k . For k infinite $r(\lambda_0, k) \rightarrow 0$.

For $V \equiv 0$ Eq. (2.43) has a unique solution, denoted by $r_{\text{free}}(\lambda_0, k) = \lambda_0/k$. The equation (2.35) is rewritten as

$$\delta(\ell_0, k) = \int_0^k \sqrt{k^2 - k'^2} \frac{d}{dk'} \left(r(\lambda_0, k') - r_{\text{free}}(\lambda_0, k') \right) dk'. \quad (2.44)$$

When V is differentiable, the derivative of the turning point with respect to k exists. We assume that the latter is locally integrable. The above equation is derived with respect to k and thus

$$\delta'_k(\ell_0, k) = \frac{d}{dk} \delta(\ell_0, k) = \int_0^k \frac{k}{\sqrt{k^2 - k'^2}} \frac{d}{dk'} \left(r(\lambda_0, k') - r_{\text{free}}(\lambda_0, k') \right) dk'. \quad (2.45)$$

Using the procedure of the inverse Abel transform [23, 33] we obtain

$$\int_0^k \frac{\delta'_k(\ell_0, u)}{\sqrt{k^2 - u^2}} du = \int_0^k \frac{d}{dk'} \left(r(\lambda_0, k') - r_{\text{free}}(\lambda_0, k') \right) dk' \int_k^{k'} du \frac{u}{\sqrt{(u^2 - k^2)(k'^2 - u^2)}} \quad (2.46)$$

using the Lebesgue-Fubini theorem [31],[32] which works when $\delta'_k(\ell_0, k)$ is Lebesgue integrable and bounded. Once the integration over u is performed we obtain

$$r(\lambda_0, k) - r_{\text{free}}(\lambda_0, k) = \frac{2}{\pi} \int_0^k \frac{\delta'_k(\ell_0, k')}{\sqrt{k^2 - k'^2}} dk', \quad (2.47)$$

provided that $r(\lambda_0, k) - r_{\text{free}}(\lambda_0, k)$ is zero for $k = 0$, which happens for finite range potential. In this latter case, for k high enough, $r(\lambda_0, k) = \lambda_0/k$ and tends to zero for k infinite.

The potential

$$V(r(\lambda_0, k)) = k^2 - \frac{\lambda_0^2}{r(\lambda_0, k)^2}$$

can be determined from (2.47) for $0 \leq r \leq R = r(\lambda_0, k_0)$, i.e. for $k \geq k_0$. However, Eq. (2.47) requires knowledge of the phase-shift $\delta(\lambda_0, k)$ for every value of k , whereas they are known only for $k \geq k_0$.

Nevertheless, these phase-shifts for $k \leq k_0$ can be determined from the potential beyond the distance R , $R = r(\lambda_0, k_0)$. Indeed, for $k \leq k_0$, (2.35) can be written as

$$\delta(\ell_0, k) = \int_{\lambda(k)}^{+\infty} \sqrt{k^2 - V(r(\lambda, k_0)) - \frac{\lambda_0^2}{r(\lambda, k_0)^2} \frac{dr(\lambda, k_0)}{d\lambda}} d\lambda - \int_{\lambda_0 k_0/k}^{+\infty} \sqrt{k^2 - \frac{k_0^2 \lambda_0^2}{\lambda^2} \frac{1}{k_0}} d\lambda. \quad (2.48)$$

Here, $\lambda(k)$ corresponds to the value of λ for which the first square root in the previous equation vanishes, i.e it is a solution of

$$k^2 - k_0^2 + \frac{\lambda^2 - \lambda_0^2}{r(\lambda, k_0)^2} = 0. \quad (2.49)$$

This implies, for $k \leq k_0$, that

$$\delta(\ell_0, k) = \int_{\lambda(k)}^{+\infty} \sqrt{k^2 - k_0^2 + \frac{\lambda^2 - \lambda_0^2}{r(\lambda, k_0)^2} \frac{dr(\lambda, k_0)}{d\lambda}} d\lambda - \int_{k_0 \lambda_0/k}^{+\infty} \sqrt{k^2 - \frac{k_0^2 \lambda_0^2}{\lambda^2} \frac{1}{k_0}} d\lambda, \quad (2.50)$$

with $r(\lambda, k_0)$ given by (2.41) in terms of the phase-shifts $\delta(\lambda, k_0)$, $\lambda \geq \lambda_0$.

3 Conclusion

In the present work, we have been concerned with a non-standard inverse scattering problem, namely with the construction of the potential from scattering data information which involves the use of phase-shifts

$$\{\delta(\ell_0, k), k \geq k_0\} \cup \{\delta(\ell, k_0), \ell \geq \ell_0\}$$

corresponding to the domain

$$\{(\ell_0, k), k \geq k_0\} \cup \{(\ell, k_0), \ell \geq \ell_0\}$$

and without requiring any extension of Newton's method for inverse scattering problems.

First, we have shown that, in the fixed- ℓ problem, the knowledge of the zeros of the regular solution the Schrödinger equation permits one to determine the potential: in fact, the zeros of the regular solution $r_n(E)$, $n \geq 1$, determine a unique potential provided the domain of the energy is such that $r_n(E)$ has values ranging from 0 to ∞ . As a second result, we have shown that piecewise constant potentials can be constructed from only one single line of zeros (say $r_{n_0}(E)$, for some fixed n_0) with values still ranging from 0 to ∞ . We also proved uniqueness theorems, stating basically that two different potentials satisfying (1.2) cannot have a common line of zeros ranging from 0 to ∞ . Furthermore, we have also considered the mixed ℓ and E problem and applied the above results to the domain $\{E \geq E_0, \ell = \ell_0\} \cup \{E = E_0, \ell \geq \ell_0\}$ for which the zeros of the regular solution for the Schrödinger equation are monotonic functions on both parts of this domain and have values still ranging from zero to infinity. This domain cannot be dealt with any extension of the Newton's method. A unique ℓ - and E -independent potential could be obtained from the set of mixed scattering data given by $\{\delta(\ell_0, k), k \geq k_0\} \cup \{\delta(\ell, k_0), \ell \geq \ell_0\}$. As the last result, we have shown that this is indeed true within the semi-classical JWKB approximation, provided there is only one turning point. Furthermore, we have shown that in Born approximation the following mixed scattering data

$$\{\delta(\ell = 0, k), k \in [k_0, +\infty[) \cup \{\delta(\ell, k_0), \ell \in \mathcal{N}\}$$

lead to an unique potential, still assumed to be ℓ - and E -independent, which is the inverse Fourier sine transform of a function deduced from the data.

To conclude, our method, which does not use an extension of Newton's method, takes advantage of a set of mixed scattering data.

A natural application concerns the discrete ambiguities (i.e. the set of different families of potentials with practically the same fixed energy phase-shift) encountered in heavy ions elastic scattering optical model analyses [11]. These latter ambiguities have been studied within the JWKB approximation in Refs. [12, 13].

Finally we point out that the flexibility of our method, based upon the properties of the zeros of the regular solution, provide a way to study inverse scattering problems which are not reducible to the $\ell = 0$ -fixed inverse scattering problem though a Liouville transformation. This latter class of problems involves the inverse scattering problem from our mixed set of data, for which there is no expected available extension of the Newton's method.

Acknowledgments

We are grateful to R.J. Lombard for many discussions and a careful reading of the manuscript. One of us (ML) is very grateful to the University of South Africa for its kind hospitality.

References

- [1] Newton R G, *Scattering Theory of Waves and Particles*, (Springer, Berlin, 1982) 2nd ed.
- [2] K. Chadani and P.C. Sabatier, *Inverse Problems in Quantum Scattering Theory* (Springer, Berlin, 1989) 2nd ed.

- [3] J.J. Loeffel, Ann. Inst. Henri Poincaré **8**, 339 (1968).
- [4] R. Ph. Boas *Entire Functions* (Academic Press, New York, 1954).
- [5] H. Grosse and A. Martin, Phys. Rep. **60**, 341 (1980).
- [6] R. Yekken, F.-Z. Ighezou and R.J. Lombard, Ann. Phys. (N.Y.) **323**, 61 (2008).
- [7] B.V. Rudyak and B.N. Zakhariev, Inverse Problems **3**, 125 (1987).
- [8] Colloque Charles Sturm, Genève University (2003).
- [9] O.H. Hald and J. R. McLaughlin, Inverse Problems **5**, 307 (1989).
- [10] C. Bender and S. Orszag, "*Advanced mathematical methods for scientists and engineers*", McGraw-Hill (1978).
- [11] D.A. Goldberg, S.M. Smith, H.G. Pugh, P.G. Roos, and N.S. Wall, Phys. Rev C **7**, 1938 (1973).
- [12] P.C. Sabatier, Phys. Rev. A **8**, 589 (1973).
- [13] M. Cuen, Ann. Physics **120**, 1 (1979).
- [14] M. Lassaut and J.-C. Wallet, in preparation.
- [15] E. Hille, *Lectures on Ordinary Differential Equations* (Addison-Wesley, Reading Mass 1969).
- [16] K. Chadan and A. Montes Lozano, Phys. Rev. **164**, 1762 (1967).
- [17] M. Lassaut, S. Y. Larsen, S. A. Sofianos and S. A. Rakityansky, J. Phys. A **34** 2007 (2001).
- [18] P. Sacks, Inverse Problems **4**, 1055 (1988). Inverse Problems **8**, 45 (1992).
- [19] I.M. Gel'fand and B.M. Levitan, Izv. Akad. Nauk SSSR, Ser. Math **15**, 309 (1951) (English translation: Am. Math. Soc. Trans. **1**, 253 (1955)
- [20] K. Chadan and M. Musette, Inverse Problems **5**, 257 (1989).
- [21] K. Chadan, R. Kobayashi and M. Musette, Inverse Problems **8**, 45 (1992).
- [22] Hochsadt H. Math. Anal. Appl. **55**, 673 (1976).
- [23] F. Riesz and B. Nagy, *Functional Analysis* (Frederick Ungar, New York, 1953).
- [24] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, vol.II, (McGraw-Hill 1953).
- [25] LL. G. Chambers, Proceedings of the American Mathematical Society **42** (1974) 140.
- [26] J. Reignier, Lettere al Nuovo Cimento **24**, 139 (1979).
- [27] P.C. Sabatier, Applicable Analysis **83**, 1077 (2004).
- [28] T. Habashy and E. Wolf, Journal of Modern Optics **41**, 1679 (1994).
- [29] E. Kujawski, Phys. Rev. C **6**, 709 (1972); Phys. Rev. C **8**, 100 (1973).
- [30] H. Fiedeldey, R. Lipperheide, K. Naidoo and S.A. Sofianos, Phys. Rev. C **30**, 434 (1984).
- [31] R. Godement *Analyse Mathématique IV* (Springer-Verlag Berlin Heidelberg 2003).
- [32] W. Rudin *Real and Complex Analysis* (McGraw-Hill 1970).
- [33] T.J. Asaki, R. Chartrand, K.R. Vixie, and B. Wohlberg, Inverse Problems **21**, 1895 (2005).